Linear Algebra I

18/12/2015, Friday, 9:00 – 11:00

You are **NOT** allowed to use any type of calculators.

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1 $(2+4+2+(4+4+4) = 20$ pts) **Linear equations**

Consider the following linear system of equations in the unknowns a, b, c, d, and e.

$$\begin{aligned} 2b + 4c + 2d + 2e &= 0 \\ 4a + 4b + 4c + 8d &= 4 \\ 8a + 2b + 10d + 2e &= 8 \\ 6a + 3b + 2c + 9d + e &= q. \end{aligned}$$

(a) Write down the augmented matrix.

(b) By performing row operations, put the augmented matrix into row echelon form.

(c) Determine all values of q so that the system is consistent.

(d) For the values of q found above,

  (i) determine the *lead* and *free* variables.

  (ii) put the augmented matrix into row *reduced* echelon form by performing row operations.

  (iii) find the solution set.

REQUIRED KNOWLEDGE: **Gauss-elimination, row operations, notions of lead/free variables.**

SOLUTION:

1a: Augmented matrix is given by:

$$\begin{bmatrix} 0 & 2 & 4 & 2 & 2 & \vdots & 0 \\ 4 & 4 & 4 & 8 & 0 & \vdots & 4 \\ 8 & 2 & 0 & 10 & 2 & \vdots & 8 \\ 6 & 3 & 2 & 9 & 1 & \vdots & q \end{bmatrix}.$$

1b:

$$\begin{bmatrix} 0 & 2 & 4 & 2 & 2 & \vdots & 0 \\ 4 & 4 & 4 & 8 & 0 & \vdots & 4 \\ 8 & 2 & 0 & 10 & 2 & \vdots & 8 \\ 6 & 3 & 2 & 9 & 1 & \vdots & q \end{bmatrix} \xrightarrow[]{\substack{\mathbf{1st = 2nd} \\ \mathbf{2nd = 1st}}} \begin{bmatrix} 4 & 4 & 4 & 8 & 0 & \vdots & 4 \\ 0 & 2 & 4 & 2 & 2 & \vdots & 0 \\ 8 & 2 & 0 & 10 & 2 & \vdots & 8 \\ 6 & 3 & 2 & 9 & 1 & \vdots & q \end{bmatrix}$$

$$\begin{bmatrix} 4 & 4 & 4 & 8 & 0 & \vdots & 4 \\ 0 & 2 & 4 & 2 & 2 & \vdots & 0 \\ 8 & 2 & 0 & 10 & 2 & \vdots & 8 \\ 6 & 3 & 2 & 9 & 1 & \vdots & q \end{bmatrix} \xrightarrow{\textbf{1st} = \frac{1}{4} \times \textbf{1st}} \begin{bmatrix} 1 & 1 & 1 & 2 & 0 & \vdots & 1 \\ 0 & 2 & 4 & 2 & 2 & \vdots & 0 \\ 8 & 2 & 0 & 10 & 2 & \vdots & 8 \\ 6 & 3 & 2 & 9 & 1 & \vdots & q \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 & 0 & \vdots & 1 \\ 0 & 2 & 4 & 2 & 2 & \vdots & 0 \\ 8 & 2 & 0 & 10 & 2 & \vdots & 8 \\ 6 & 3 & 2 & 9 & 1 & \vdots & q \end{bmatrix} \xrightarrow[]{\substack{\textbf{3rd} = \textbf{3rd} - 8 \times \textbf{1st} \\ \textbf{4th} = \textbf{4th} - 6 \times \textbf{1st}}} \begin{bmatrix} 1 & 1 & 1 & 2 & 0 & \vdots & 1 \\ 0 & 2 & 4 & 2 & 2 & \vdots & 0 \\ 0 & -6 & -8 & -6 & 2 & \vdots & 0 \\ 0 & -3 & -4 & -3 & 1 & \vdots & q-6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 & 0 & \vdots & 1 \\ 0 & 2 & 4 & 2 & 2 & \vdots & 0 \\ 0 & -6 & -8 & -6 & 2 & \vdots & 0 \\ 0 & -3 & -4 & -3 & 1 & \vdots & q-6 \end{bmatrix} \xrightarrow{\textbf{2nd} = \frac{1}{2} \times \textbf{2nd}} \begin{bmatrix} 1 & 1 & 1 & 2 & 0 & \vdots & 1 \\ 0 & 1 & 2 & 1 & 1 & \vdots & 0 \\ 0 & -6 & -8 & -6 & 2 & \vdots & 0 \\ 0 & -3 & -4 & -3 & 1 & \vdots & q-6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 & 0 & \vdots & 1 \\ 0 & 1 & 2 & 1 & 1 & \vdots & 0 \\ 0 & -6 & -8 & -6 & 2 & \vdots & 0 \\ 0 & -3 & -4 & -3 & 1 & \vdots & q-6 \end{bmatrix} \xrightarrow[]{\substack{\textbf{3rd} = \textbf{3rd} + 6 \times \textbf{2nd} \\ \textbf{4th} = \textbf{4th} + 3 \times \textbf{2nd}}} \begin{bmatrix} 1 & 1 & 1 & 2 & 0 & \vdots & 1 \\ 0 & 1 & 2 & 1 & 1 & \vdots & 0 \\ 0 & 0 & 4 & 0 & 8 & \vdots & 0 \\ 0 & 0 & 2 & 0 & 4 & \vdots & q-6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 & 0 & \vdots & 1 \\ 0 & 1 & 2 & 1 & 1 & \vdots & 0 \\ 0 & 0 & 4 & 0 & 8 & \vdots & 0 \\ 0 & 0 & 2 & 0 & 4 & \vdots & q-6 \end{bmatrix} \xrightarrow{\textbf{3rd} = \frac{1}{4} \times \textbf{3rd}} \begin{bmatrix} 1 & 1 & 1 & 2 & 0 & \vdots & 1 \\ 0 & 1 & 2 & 1 & 1 & \vdots & 0 \\ 0 & 0 & 1 & 0 & 2 & \vdots & 0 \\ 0 & 0 & 2 & 0 & 4 & \vdots & q-6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 & 0 & \vdots & 1 \\ 0 & 1 & 2 & 1 & 1 & \vdots & 0 \\ 0 & 0 & 1 & 0 & 2 & \vdots & 0 \\ 0 & 0 & 2 & 0 & 4 & \vdots & q-6 \end{bmatrix} \xrightarrow{\textbf{4th} = \textbf{4th} - 2 \times \textbf{3rd}} \begin{bmatrix} 1 & 1 & 1 & 2 & 0 & \vdots & 1 \\ 0 & 1 & 2 & 1 & 1 & \vdots & 0 \\ 0 & 0 & 1 & 0 & 2 & \vdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \vdots & q-6 \end{bmatrix}$$

1c: It is consistent if and only if $q = 6$.

1d: If $q = 6$, then we have

$$\begin{bmatrix} 1 & 1 & 1 & 2 & 0 & \vdots & 1 \\ 0 & 1 & 2 & 1 & 1 & \vdots & 0 \\ 0 & 0 & 1 & 0 & 2 & \vdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 \end{bmatrix}.$$

(i): Lead variables are a, b, and c whereas d and e are free variables.

(ii):

$$\left[\begin{array}{ccccccc} 1 & 1 & 1 & 2 & 0 & \vdots & 1 \\ 0 & 1 & 2 & 1 & 1 & \vdots & 0 \\ 0 & 0 & 1 & 0 & 2 & \vdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 \end{array}\right] \xrightarrow[\textbf{1st} = \textbf{1st} - \textbf{3rd}]{\textbf{2nd} = \textbf{2nd} - 2\times\textbf{3rd}} \left[\begin{array}{ccccccc} 1 & 1 & 0 & 2 & -2 & \vdots & 1 \\ 0 & 1 & 0 & 1 & -3 & \vdots & 0 \\ 0 & 0 & 1 & 0 & 2 & \vdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 \end{array}\right]$$

$$\left[\begin{array}{ccccccc} 1 & 1 & 0 & 2 & -2 & \vdots & 1 \\ 0 & 1 & 0 & 1 & -3 & \vdots & 0 \\ 0 & 0 & 1 & 0 & 2 & \vdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 \end{array}\right] \xrightarrow{\textbf{1st} = \textbf{1st} - \textbf{2nd}} \left[\begin{array}{ccccccc} 1 & 0 & 0 & 1 & 1 & \vdots & 1 \\ 0 & 1 & 0 & 1 & -3 & \vdots & 0 \\ 0 & 0 & 1 & 0 & 2 & \vdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 \end{array}\right]$$

(iii): The general solution is given by

$$a = 1 - d - e$$
$$b = -d + 3e$$
$$c = -2e$$

where d and e are free variables.

Find all values of a, b, and c such that the matrix

$$\begin{bmatrix} a^2 & (a+1)^2 & (a+2)^2 \\ b^2 & (b+1)^2 & (b+2)^2 \\ c^2 & (c+1)^2 & (c+2)^2 \end{bmatrix}.$$

is nonsingular.

REQUIRED KNOWLEDGE: **Determinants, nonsingular matrices.**

SOLUTION:

2a: First we compute the determinant of this matrix. By applying row and column operations, we get

$$\det\Big(\begin{bmatrix} a^2 & (a+1)^2 & (a+2)^2 \\ b^2 & (b+1)^2 & (b+2)^2 \\ c^2 & (c+1)^2 & (c+2)^2 \end{bmatrix}\Big) = \det\Big(\begin{bmatrix} a^2 & 2a+1 & 4a+4 \\ b^2 & 2b+1 & 4b+4 \\ c^2 & 2c+1 & 4c+4 \end{bmatrix}\Big) \quad \left\{\begin{matrix}\text{column operations} \\ \textbf{2nd} = \textbf{2nd} - \textbf{1st} \\ \textbf{3rd} = \textbf{3rd} - \textbf{1st}\end{matrix}\right\}$$

$$= 4\det\Big(\begin{bmatrix} a^2 & 2a+1 & a+1 \\ b^2 & 2b+1 & b+1 \\ c^2 & 2c+1 & c+1 \end{bmatrix}\Big) \quad \left\{\begin{matrix}\text{column operation} \\ \textbf{3rd} = \frac{1}{4} \times \textbf{3rd}\end{matrix}\right\}$$

$$= 4\det\Big(\begin{bmatrix} a^2 & a & a+1 \\ b^2 & b & b+1 \\ c^2 & c & c+1 \end{bmatrix}\Big) \quad \left\{\begin{matrix}\text{column operations} \\ \textbf{2nd} = \textbf{2nd} - \textbf{3rd}\end{matrix}\right\}$$

$$= 4\det\Big(\begin{bmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{bmatrix}\Big) \quad \left\{\begin{matrix}\text{column operation} \\ \textbf{3rd} = \textbf{3rd} - \textbf{2nd}\end{matrix}\right\}$$

$$= 4\det\Big(\begin{bmatrix} 0 & 0 & 1 \\ b^2-a^2 & b-a & 1 \\ c^2-a^2 & c-a & 1 \end{bmatrix}\Big) \quad \left\{\begin{matrix}\text{column operations} \\ \textbf{2nd} = \textbf{2nd} - a \times \textbf{3rd} \\ \textbf{1st} = \textbf{1st} - a^2 \times \textbf{3rd}\end{matrix}\right\}$$

$$= 4\det\Big(\begin{bmatrix} b^2-a^2 & b-a \\ c^2-a^2 & c-a \end{bmatrix}\Big) \quad \{\textbf{Cofactor exp. w.r.t. 1st row}\}$$

$$= 4(b-a)(c-a)\det\Big(\begin{bmatrix} b+a & 1 \\ c+a & 1 \end{bmatrix}\Big) \quad \left\{\begin{matrix}\text{row operations} \\ \textbf{1st} = \frac{1}{b-a} \times \textbf{1st} \\ \textbf{2nd} = \frac{1}{c-a} \times \textbf{2nd}\end{matrix}\right\}$$

$$= 4(b-a)(c-a)(b-c)$$

A square matrix is nonsingular if and only if its determinant is nonzero. Therefore, the matrix we look at is nonsingular if and only if $a \neq b$ and $b \neq c$ and $c \neq a$.

Let x, y be n vectors. Consider the matrix

$$M = \begin{bmatrix} I_n & x \\ y^T & 1 \end{bmatrix}.$$

(a) Show that

$$\det(M) = \det(I_n - xy^T) = 1 - y^T x.$$

(b) Assume that $y^T x \neq 1$ and find the inverse of M.

REQUIRED KNOWLEDGE: **Partitioned matrices, nonsingular matrices, and inverse and the fact that there is no ethical consumption under capitalism**

SOLUTION:

3a: Note that

$$\begin{bmatrix} I_n & x \\ y^T & 1 \end{bmatrix} \begin{bmatrix} I_n & 0_{n\times 1} \\ -y^T & 1 \end{bmatrix} = \begin{bmatrix} I_n - xy^T & x \\ 0 & 1 \end{bmatrix}.$$

This corresponds to application of block column operations. Hence, we have

$$\det(\begin{bmatrix} I_n & x \\ y^T & 1 \end{bmatrix}) \underbrace{\det(\begin{bmatrix} I_n & 0_{n\times 1} \\ -y^T & 1 \end{bmatrix})}_{=1} = \underbrace{\det(\begin{bmatrix} I_n - xy^T & x \\ 0 & 1 \end{bmatrix})}_{=\det(I_n - xy^T)}.$$

Therefore,

$$\det(\begin{bmatrix} I_n & x \\ y^T & 1 \end{bmatrix}) = \det(I_n - xy^T).$$

Similarly, note that

$$\begin{bmatrix} I_n & 0_{n\times 1} \\ -y^T & 1 \end{bmatrix} \begin{bmatrix} I_n & x \\ y^T & 1 \end{bmatrix} = \begin{bmatrix} I_n & x \\ 0 & 1 - y^T x \end{bmatrix}.$$

This results in

$$\underbrace{\det(\begin{bmatrix} I_n & 0_{n\times 1} \\ -y^T & 1 \end{bmatrix})}_{=1} \det(\begin{bmatrix} I_n & x \\ y^T & 1 \end{bmatrix}) = \underbrace{\det(\begin{bmatrix} I_n & x \\ 0 & 1 - y^T x \end{bmatrix})}_{=1 - y^T x}.$$

Consequently we obtain

$$\det(\begin{bmatrix} I_n & x \\ y^T & 1 \end{bmatrix}) = 1 - y^T x.$$

3b: If $y^T x \neq 1$, then $\det(M) \neq 0$. In other words, M is nonsingular. To find its inverse, we first take

$$M^{-1} = \begin{bmatrix} P & q \\ r^T & s \end{bmatrix}$$

where P is an $n \times n$ matrix, $q, r \in \mathbb{R}^n$, and $s \in \mathbb{R}$. Note that

$$\begin{bmatrix} I_n & x \\ y^T & 1 \end{bmatrix} \begin{bmatrix} P & q \\ r^T & s \end{bmatrix} = \begin{bmatrix} I_n & 0_{n\times 1} \\ 0_{1\times n} & 1 \end{bmatrix}.$$

This gives us

$$\begin{aligned} I_n &= P + xr^T \\ 0_{n\times 1} &= q + sx \\ 0_{1\times n} &= y^T P + r^T \\ 1 &= y^T q + s. \end{aligned}$$

Solving r^T from the third and substituting it into the first result in

$$I_n = P - xy^T P = (I_n - xy^T)P.$$

Hence, we obtain

$$P = (I_n - xy^T)^{-1}$$

and

$$r^T = -y^T(I_n - xy^T)^{-1}.$$

Now, we first solve q from the second and substitute into the forth. This results in

$$1 = -sy^T x + s = s(1 - y^T x)$$

. Therefore,

$$s = (1 - y^T x)^{-1}$$

and

$$q = -(1 - y^T x)^{-1} x.$$

Consequently, we get

$$M^{-1} = \begin{bmatrix} (I_n - xy^T)^{-1} & -(1 - y^T x)^{-1} x \\ -y^T(I_n - xy^T)^{-1} & (1 - y^T x)^{-1} \end{bmatrix}.$$

Consider the vector space $\mathbb{R}^{n\times n}$. Let $A \in \mathbb{R}^{n\times n}$.

(a) Let
$$S_1 = \{X \in \mathbb{R}^{n\times n} \mid AX + XA = 0_{n\times n}\}.$$
Is S_1 a subspace? Justify your answer.

(b) Let
$$S_2 = \{X \in \mathbb{R}^{n\times n} \mid AX + XA = A\}.$$
Is S_2 a subspace? Justify your answer.

REQUIRED KNOWLEDGE: **Subspaces.**

SOLUTION:

4a: We begin with observing that $0_{n\times n} \in S_1$. Hence, S_1 is nonempty. Let α be a scalar and $X \in S_1$. Note that
$$A(\alpha X) + (\alpha X)A = \alpha(AX + XA) = 0_{n\times n}.$$
As such, $\alpha X \in S_1$, that is S_1 is closed under scalar multiplication.

Now, let $X, Y \in S_1$. Note that
$$A(X + Y) + (X + Y)A = AX + XA + AY + YA = 0_{n\times n} + 0_{n\times n} = 0_{n\times n}.$$
Thus, $X + Y \in S_1$, that is S_1 is closed under vector addition.

Consequently, S_1 is a subspace.

4b: First, we note that $\frac{1}{2}I_n \in S_2$ since
$$A(\frac{1}{2}I_n) + (\frac{1}{2}I_n)A = A.$$
As such, S_2 is nonempty.

Let α be a scalar and $X \in S_2$. Note that
$$A(\alpha X) + (\alpha X)A = \alpha(AX + XA) = \alpha A.$$
This means that $\alpha X \in S_2$ if and only if $\alpha A = A$ for all scalars α. In other words, $\alpha X \in S_2$ if and only if $A = 0_{n\times n}$.

Now, let $X, Y \in S_2$. Note that
$$A(X + Y) + (X + Y)A = AX + XA + AY + YA = 2A.$$
As such, $X + Y \in S_2$ if and only if $2A = A$, that is $A = 0_{n\times n}$.

Consequently, S_2 is a subspace if and only if $A = 0_{n\times n}$.